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LemmaCalc:

Quick Theory Exploration for Algebraic Data Types via Program Transformations

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published at iFM 2025, best paper candidate

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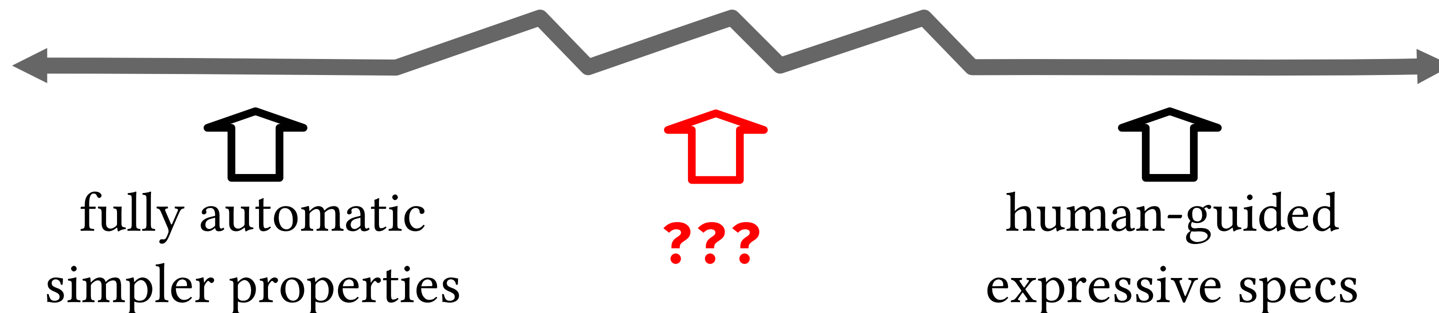


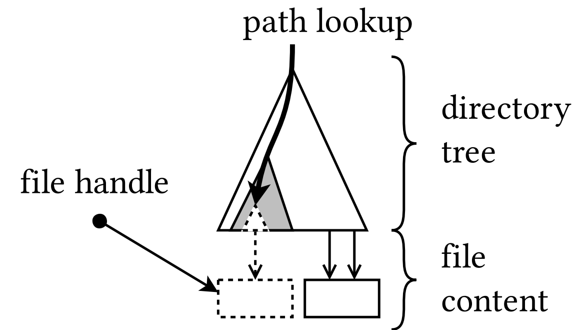
Motivation: Automated, Interactive Proofs

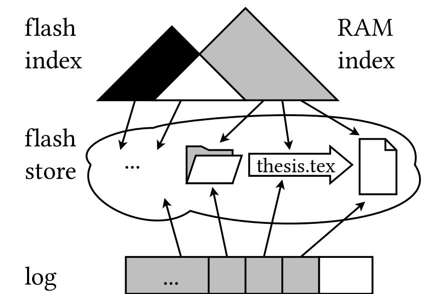
A typical verification case-study

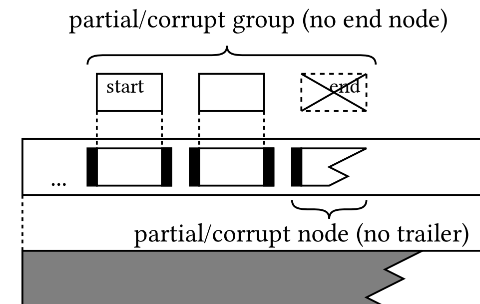
- custom data types & function definitions
- expressivene logic (recursion, quantifiers)
- incremental development

Ex 1: kiv.isse.de/projects-v8/flash.html

Ex 2: VerifyThis h-Index challenge







Flashix [Ernst+]

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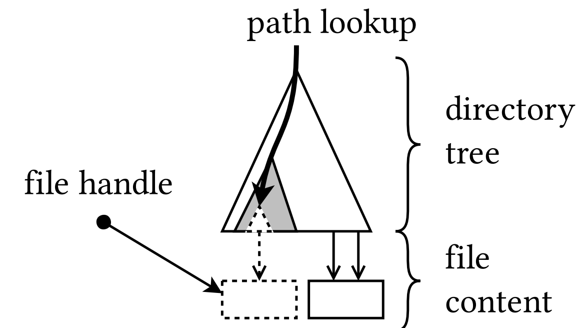

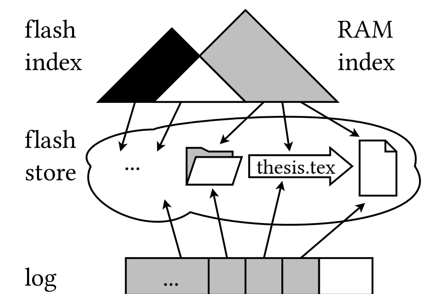

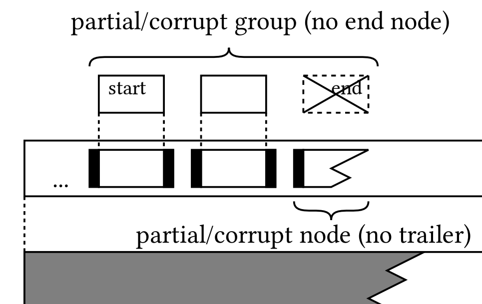

Typical Questions:

- `get(update( ... )) = ???`
- `abs(impl( ... ))   = ???`

Flashix [Ernst+]

Motivation: Automated, Interactive Proofs

A typical verification case-study

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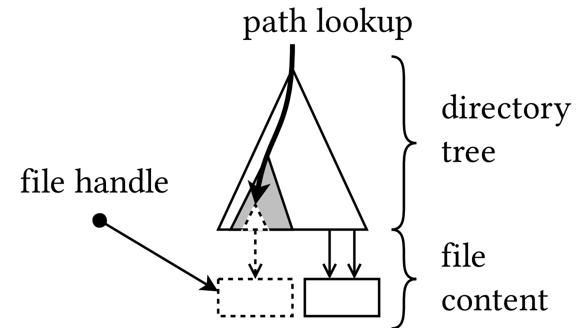
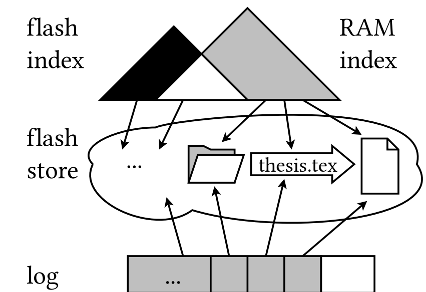
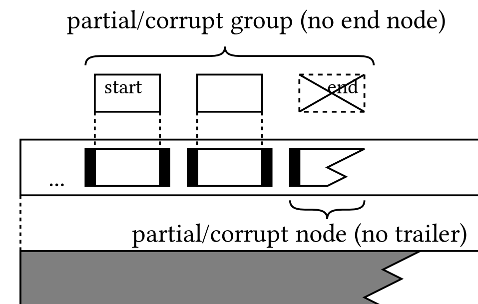
- expressivene logic (recursion, quantifiers)
- incremental development





Bottleneck: formulating & proving lemmas

- break down proof obligations
- automate recurring proof steps



Flashix [Ernst+]

Goal: Theory Exploration

Input

```
data Nat         :=  0 | 1+ Nat
data List<a>     := [] | a :: List<a>

length([])       := 0
length(x::xs)    := 1 + length(xs)

      [] ++ ys   := ys
(x::xs) ++ ys    := x::(xs ++ ys)
```

inductive types

recursive functions

Goal: Theory Exploration

Input

```
data Nat         :=  0 | 1+ Nat
data List<a>     := [] | a :: List<a>

length([])       := 0
length(x::xs)    := 1 + length(xs)

     [] ++ ys    := ys
(x::xs) ++ ys    := x::(xs ++ ys)
```

inductive types

recursive functions

Output

```
lemma length(xs ++ ys)
          = length(xs) + length(ys)
 ...
```

“useful/ intersting” lemmas

Classic Approach: Term Enumeration

[Claessen, Hughes, Johansson+, Singher & Itzhaky, ...]

Algorithm

```
repeat
    sample phi: Bool with |phi| < N
    if theory ∪ lemmas proves phi
        lemmas := lemmas ∪ {phi}
```

Classic Approach: Term Enumeration

[Claessen, Hughes, Johansson+, Singher & Itzhaky, …]

Algorithm

```
repeat
    sample phi: Bool with |phi| < N
    if theory ∪ lemmas proves phi
        lemmas := lemmas ∪ {phi}
```

"complete" relative to search space and prover

In practice

- further restrictions: shape, few duplicate vars, …
- relevance filter: requires induction, …
- testing (QuickCheck) vs. inductive proofs (HipSpec, TheSy)

Status Quo

```
length(xs) ?= 0
```

...

Lemma Calc

Exploiting Structure during Lemma Search

Status Quo

```
length(xs) ?= 0
  …
length(xs ++ ys) ?= 0
length(xs ++ ys) ?= length([])
length(xs ++ ys) ?= length(xs)
  …
```

Lemma Calc

Status Quo

```
length(xs) ?= 0
   …
length(xs ++ ys) ?= 0
length(xs ++ ys) ?= length([])
length(xs ++ ys) ?= length(xs)
   …
length(xs ++ ys) ?=
  length(xs) + length(ys)
   …
```

Lemma Calc

Combinatorial Search Spaces (Table 1)

		baseline enumerator statistics					TheSy	
benchmark	$\|F\|$	candidates	true	$\|\Lambda\|$	?	time	last	killed
`nat`	8	1 131 799	501	32	1759	6:50:00	26:38:14	>26h
`list`	18	319 019	408	32	522	1:48:22	10:55:14	>21h
`tree`	11	123 178	130	20	38	11:25	16:47	
`append`	5	15 058	133	22	5	02:03	04:32	
`filter`	6	398	2	5	16	02:11	00:02	
`length`	5	7 066	558	12	1	01:59	00:00	
`map`	8	34 726	103	13	35	07:18	37:33	>11h
`remove`	7	32 302	117	14	13	22:11	13:01	>11h
`reverse`	4	127 926	427	22	1	03:29	00:02	
`rotate`	6	12 784	124	20	43	08:50	6:54:22	>11h
`runlength`	7	68 311	182	23	847	†1:12:12	00:40	>11h

long runtimes

exponential in |definitions|

“interesting” lemmas are rare

productivity is very sporadic

Example Lemmas over Lists and Trees


```
length(filter(p, xs)) ≡ countif(p, xs)
take(n, map(f, xs))   ≡ map(f, take(n, xs))
rev(rev(xs))          ≡ xs
sum(ys ++ zs)         ≡ sum(ys) + sum(zs)
length(elems(t))      ≡ size(t)
...
```

Example Lemmas over Lists and Trees


```
length(filter(p, xs)) = countif(p, xs)
take(n, map(f, xs))   = map(f, take(n, xs))
rev(rev(xs))          = xs
sum(ys ++ zs)         = sum(ys) + sum(zs)
length(elems(t))      = size(t)
...
```

```
...
rev(rev(xs))
  = take(length(xs), snoc(xs, zero))
```

lemmas from term enumeration are not necessarily "useful"

Contribution

10.5281/zenodo.16932462

LemmaCalc: an approach for theory exploration

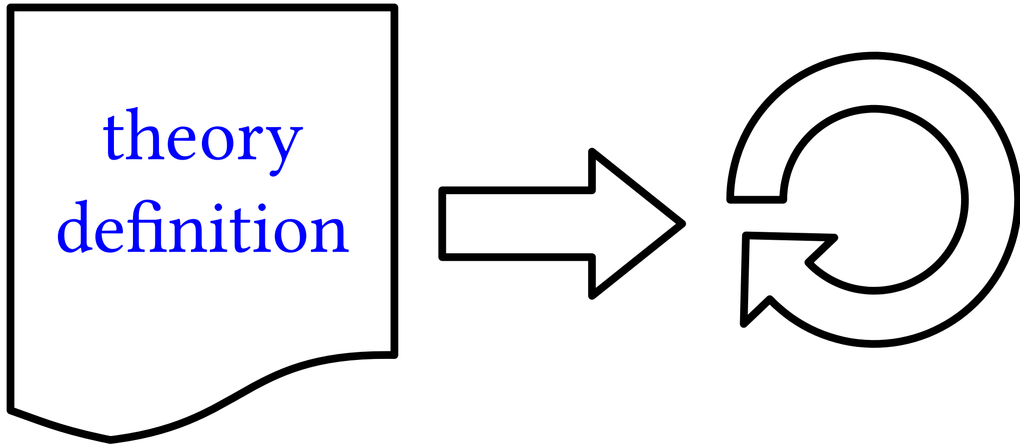
- **quickly** explores a **structured** search space
- lemmas are **intuitive** for humans
- lemmas tend to be **useful** for automation

Focus: **equations** (associative/distributive laws)

Main method: integrate **program transformations** and **deduction**

Evaluation against enumerative theory exploration

- RQ1: relative explanatory strength of the sets of lemmas generated?
- RQ2: impact of the search space on lemma synthesis time?

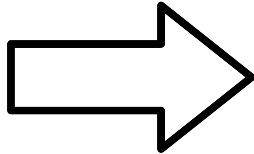

theory
definition

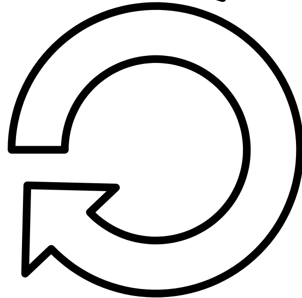
exploration by program transformations

fusion: `f(x, g(y)) = fg(x, y)`

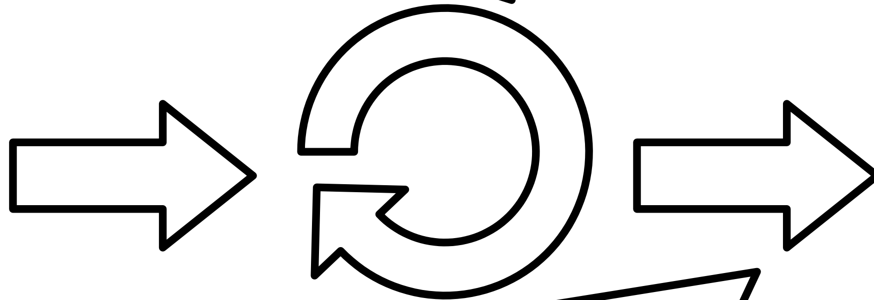
accumulator removal: `h(x,y) = h'(x) ⊕ e(y)`

theory definition





generates **synthetic functions** to represent intermediate results



exploration by program transformations
fusion: f(x, g(y)) = fg(x, y)
accumulator removal: h(x,y) = h'(x) ⊕ e(y)
theory definition
extracted lemmas
recognition principles
constant functions: f(x) = c
identity functions: f(x) = x
structural equivalence: f(x) = g(π(x))

Exploiting Structure during Lemma Search

Term enumeration

```
length(xs) ≠ 0
  …
length(xs ++ ys) ≠ 0
length(xs ++ ys) ≠ length([])
length(xs ++ ys) ≠ length(xs)
  …
length(xs ++ ys) ==
  length(xs) + length(ys)
  …
```

Lemma Calc

- **Explore**

```
f1 := FUSE(length(_ ++ _))
```

compiler transformation
[Turchin, Wadler, SPJ, Meijer, …, Burstall & Darlington, Sonnex, deAngelis]

Exploiting Structure during Lemma Search

Term enumeration

```
length(xs) ≠ 0
   ...
length(xs ++ ys) ≠ 0
length(xs ++ ys) ≠ length([])
length(xs ++ ys) ≠ length(xs)
   ...
length(xs ++ ys) ==
  length(xs) + length(ys)
   ...
```

Lemma Calc

- **Explore**

  `f1 := FUSE(length(_ ++ _))`

  ...

- **Explore**

  `f2 := FUSE(length(_) + _)`

Exploiting Structure during Lemma Search

Term enumeration

```
length(xs) ≠ 0
   …
length(xs ++ ys) ≠ 0
length(xs ++ ys) ≠ length([])
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   …
length(xs ++ ys) ==
  length(xs) + length(ys)
   …
```

Lemma Calc

- **Explore**

```
f1 := FUSE(length(_ ++ _))
```

…

- **Explore**

```
f2 := FUSE(length(_) + _)
```

- **Short-Cut** (rec. unification)

```
UNIFY(f1(x,y) ?= f2(x',y'))
⇝ x' = x and y' = length(y)
```

Exploiting Structure during Lemma Search

new foundations
for symbolic reasoning

lots of concrete technical
challenges and opportunities

Lemma Calc

- **Explore**

```
f1 := FUSE(length(_ ++ _))
```

…

- **Explore**

```
f2 := FUSE(length(_) + _)
```

- **Short-Cut** (rec. unification)

```
UNIFY(f1(x,y) ?= f2(x',y'))
⇝ x' = x and y' = length(y)
```

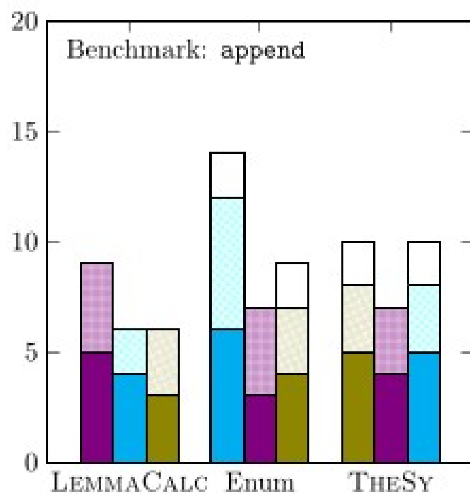
Evaluation

RQ1: What is the **relative explanatory strength** of the sets of lemmas generated by the different methods?

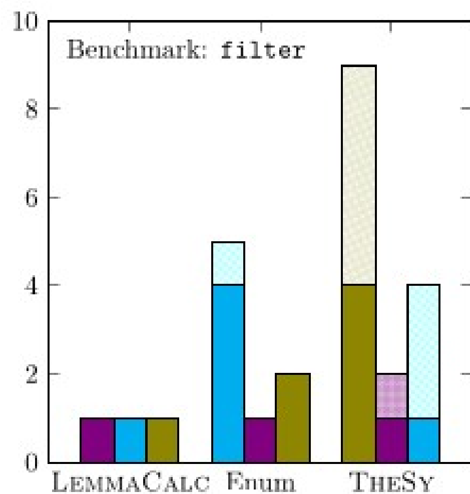
- comparable, but complementary strengths and weaknesses

RQ2: What is the impact of the search space on lemma **synthesis time**?

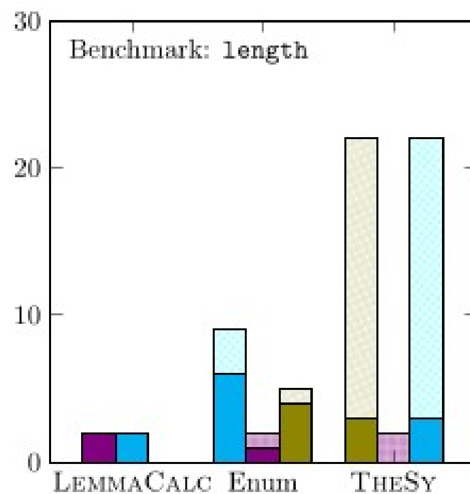
- LemmaCalc is much faster but misses some results



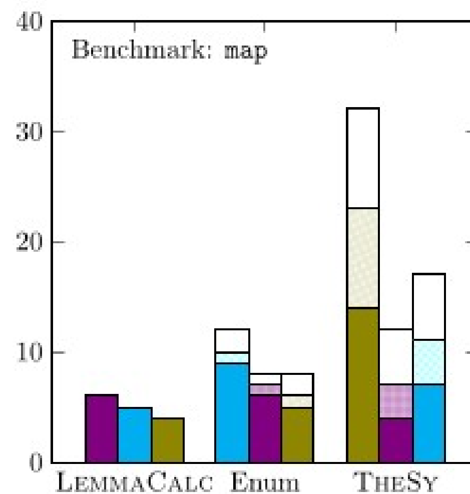
Benchmark: append
LemmaCalc
Enum
TheSy



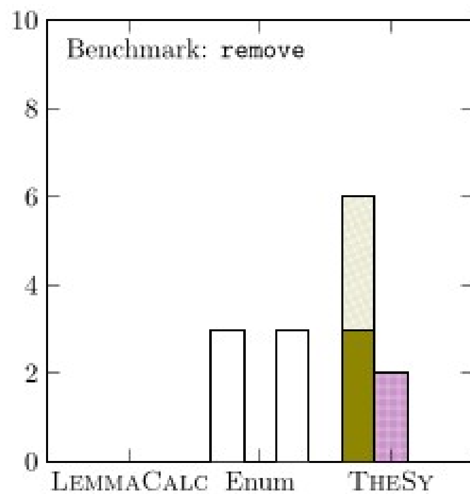
Benchmark: filter
LemmaCalc
Enum
TheSy



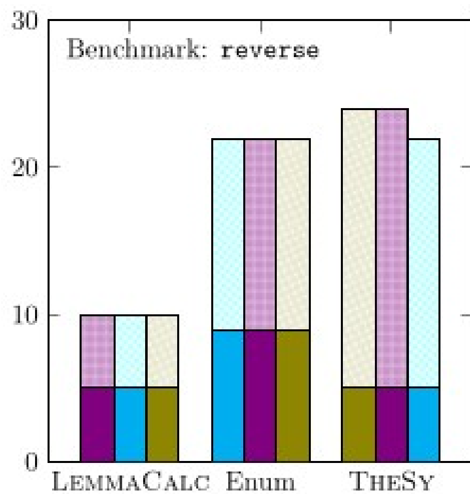
Benchmark: length
LemmaCalc
Enum
TheSy



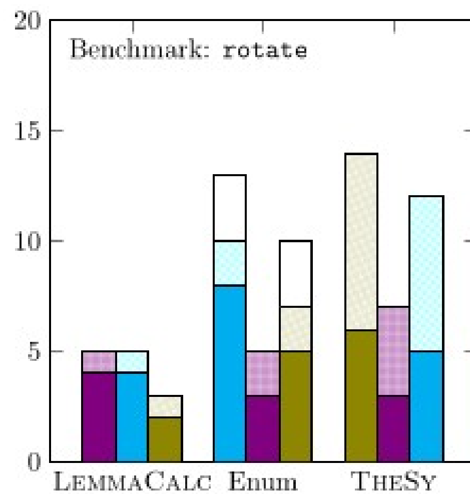
Benchmark: map
LemmaCalc
Enum
TheSy



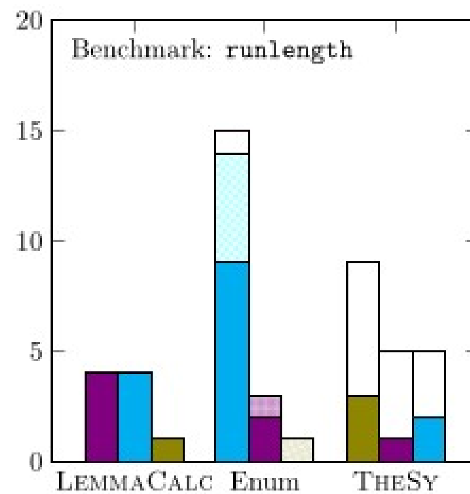
Benchmark: remove
LemmaCalc
Enum
TheSy



Benchmark: reverse
LemmaCalc
Enum
TheSy



Benchmark: rotate
LemmaCalc
Enum
TheSy

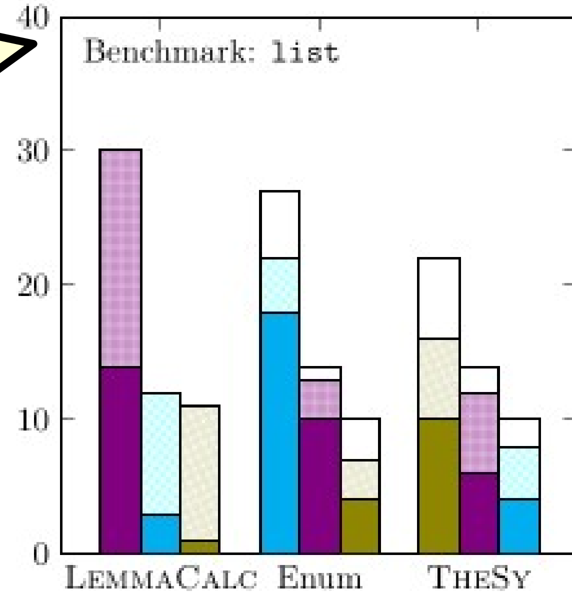


Benchmark: runlength
LemmaCalc
Enum
TheSy

Results: Number of Lemmas

LemmaCalc
Baseline Enumerator
TheSy [Singher, Itzhaky]

"large" theory:
18 functions

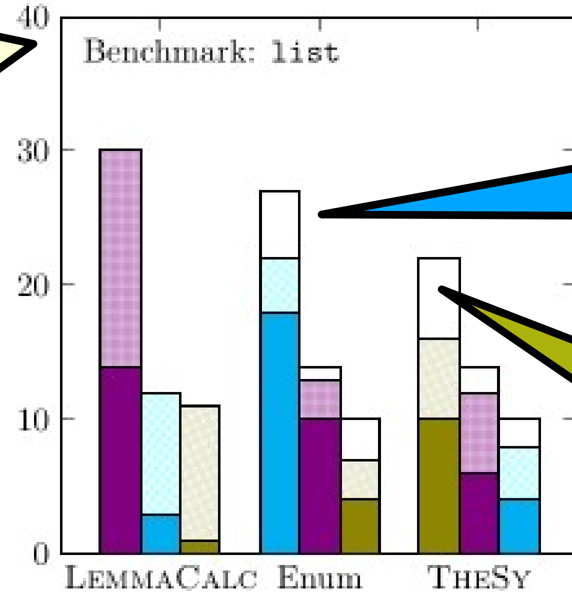


~10s >24h each

Results: Number of Lemmas

LemmaCalc
Baseline Enumerator
TheSy [Singher, Itzhaky]

“large” theory:
18 functions



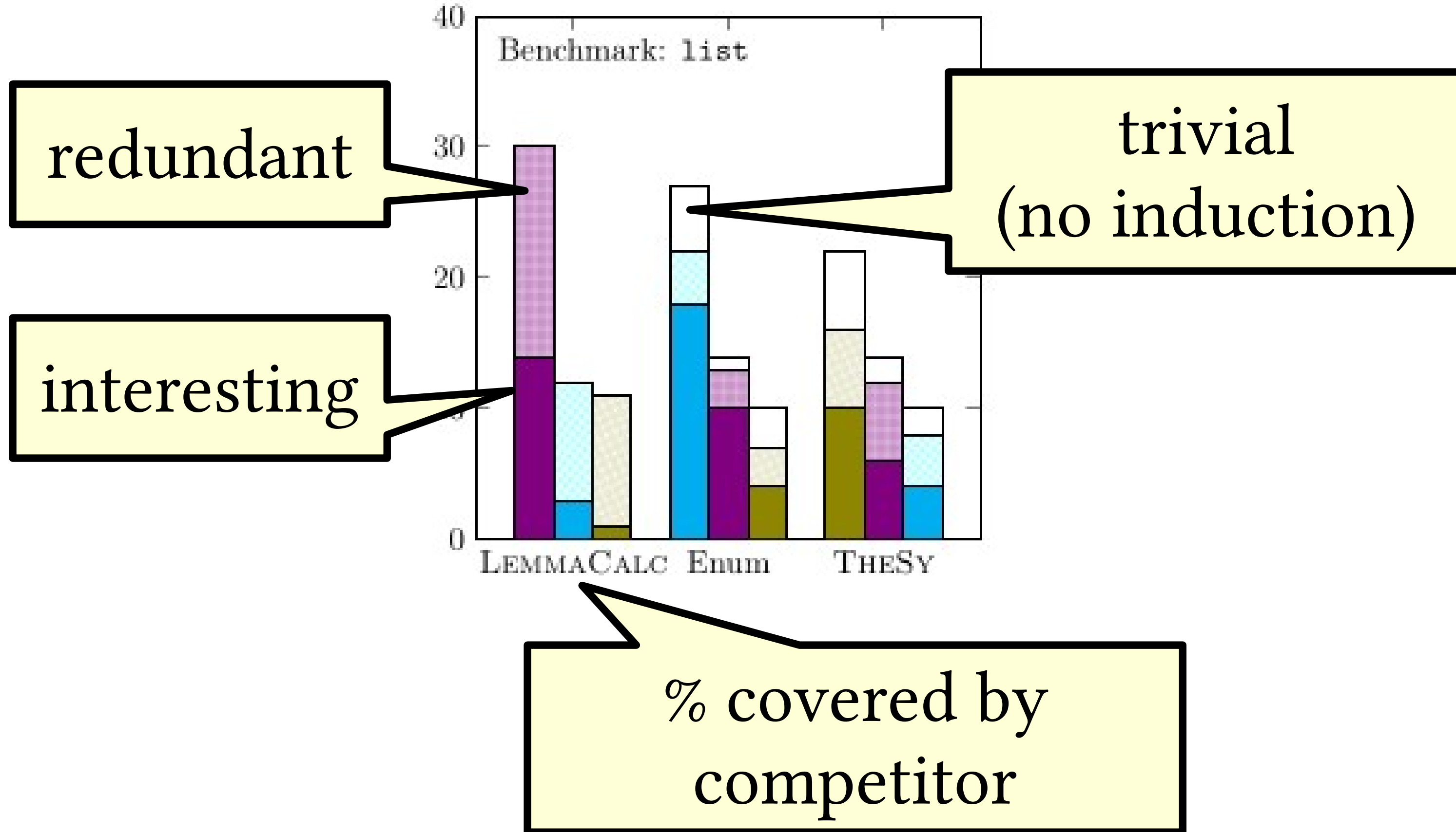
> 300K candidates
`f(x, g(y)) = ?`

with conditional
lemmas

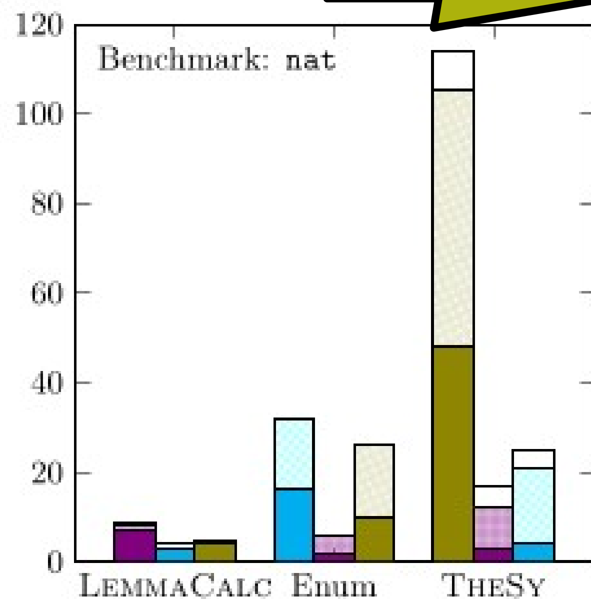
~10s >24h each

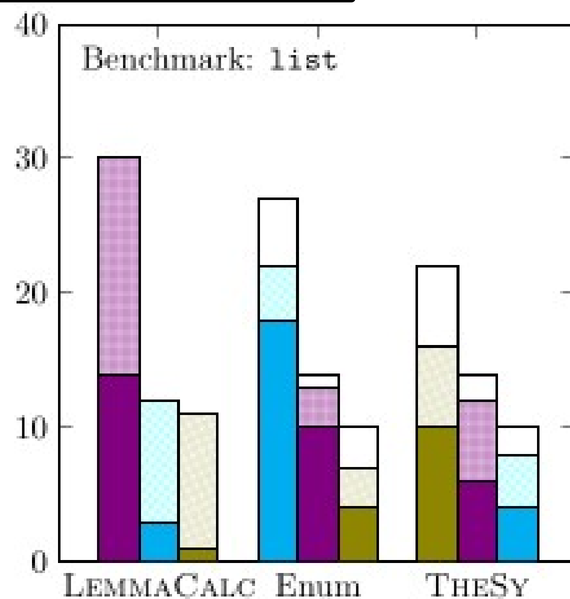
Results: Number of Lemmas

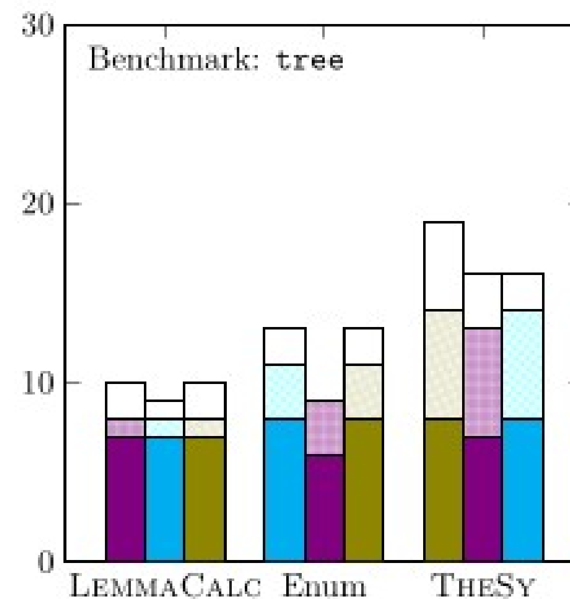
LemmaCalc
Baseline Enumerator
TheSy [Singher, Itzhaky]



many
conditional lemmas







(almost) identical
non-redundant sets

Applications beyond Formal Proof

`add(E e)`	Ensures that this collection contains the specified element (optional operation).
`addAll(Collection<? extends E> c)`	Adds all of the elements in the specified collection to this collection (optional operation).
`clear()`	Removes all of the elements from this collection (optional operation).
`contains(Object o)`	Returns `true` if this collection contains the specified element.
`containsAll(Collection<?> c)`	Returns `true` if this collection contains all of the elements in the specified collection.

Methods are specified with respect to each other!
Can we **extract** PBT tests like this?

```
@ForAll c, e: c.add(e); assert(c.contains(e));
```

https://docs.oracle.com/en/java/javase/17/docs/api/java.base/java/util/Collection.html

Related Work

- Buchberger, Claessen, Hughes, Johansson
  pioneered theory exploration (tools & larger case studies)

- Sonnex, deAngelis
  program transformations are effective proof methods
- Hamilton & Poítin, Klyuchnikov & Romanenko
  suggest fusion for lemma inference (= low hanging fruits)
- Giesl: deaccumulation schemes (not implemented)
- lots of work in the PL community (see paper)

Contribution: LemmaCalc

- **quickly** explores a productive search space
- lemmas tend to be **intuitive** and **useful**
- integrate **transformations** and **deduction**

- challenge the conventional wisdom that expressiveness $\neq$ automation
- aim for automation that can introduce new concepts
- create new foundations for symbolic reasoning

Appendix

What's a Lemma?

one way to present
a computation

$$
\begin{aligned}
&(x - y)^2 \\
&\quad = (x - y)(x - y) \\
&\quad = x^2 - xy - yx + y^2 \\
&\quad = x^2 - xy - xy + y^2 \\
&\quad = x^2 - 2xy + y^2
\end{aligned}
$$

another way to present
a computation with the same result

What's a Lemma?

$$
\begin{aligned}
&(x - y)^2 \\
&\quad = (x - y)(x - y) \\
&\quad = x^2 - xy - yx + y^2 \\
&\quad = x^2 - xy - xy + y^2 \\
&\quad = x^2 - 2xy + y^2
\end{aligned}
$$

definition of $(_)^2$

use lemma $xy = yx$

What's a Lemma?

one way to present
a computation

$$\texttt{rev}(xs_1 \mathbin{+\!\!+} xs_2)$$

$$= \textbf{???}$$

$$= \texttt{rev}(xs_2) \mathbin{+\!\!+} \texttt{rev}(xs_1)$$

another way to present
a computation with the same result

Calculating with Functions

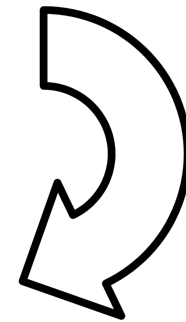
[Bird, Burstall & Darlington, Meijer, …]

$$
\begin{aligned}
&\texttt{rev}(\texttt{xs}_1 \mathbin{+\!\!+} \texttt{xs}_2)\\
&\quad = \texttt{rev}{+\!\!+}(\texttt{xs}_1, \texttt{xs}_2)\\
&\quad = \texttt{rev}{+\!\!+}'(\texttt{xs}_1) \oplus^{?} e^{?}(\texttt{xs}_2)\\
&\quad = \texttt{rev}(\texttt{xs}_2) \mathbin{+\!\!+} \texttt{rev}{+\!\!+}'(\texttt{xs}_1)\\
&\quad = \texttt{rev}(\texttt{xs}_2) \mathbin{+\!\!+} \texttt{rev}(\texttt{xs}_1)
\end{aligned}
$$

Calculating with Functions

[Bird, Burstall & Darlington, Meijer, ...]

$$\mathtt{rev}(\mathtt{xs}_1 \mathbin{+\!\!+} \mathtt{xs}_2)$$
$$= \mathtt{rev{+\!\!+}}(\mathtt{xs}_1, \mathtt{xs}_2)$$
$$= \mathtt{rev{+\!\!+}'}(\mathtt{xs}_1) \oplus^{?} \mathtt{e}^{?}(\mathtt{xs}_2)$$
$$= \mathtt{rev}(\mathtt{xs}_2) \mathbin{+\!\!+} \mathtt{rev{+\!\!+}'}(\mathtt{xs}_1)$$
$$= \mathtt{rev}(\mathtt{xs}_2) \mathbin{+\!\!+} \mathtt{rev}(\mathtt{xs}_1)$$



unblocks

$$\mathtt{rev}(\mathtt{rev}(\mathtt{xs}))$$
$$= \ldots = \mathtt{id}(\mathtt{xs}) = \mathtt{xs}$$

Calculating with Functions

[Bird, Burstall & Darlington, Meijer, ...]

```
rev(xs₁ ++ xs₂)
    = rev++(xs₁, xs₂)
```

[Wadler, SPJ, Turchin, ...]

Calculating with Functions

[Bird, Burstall & Darlington, Meijer, ...]

$$\mathtt{rev}(xs_1 +\!\!+\ xs_2)$$
$$= \mathtt{rev}{+\!\!+}(xs_1,\ xs_2)$$
$$= \mathbf{rev{+\!\!+}'}(\mathbf{xs_1})\ \oplus^?\ e^?(xs_2)$$

template for accumulator removal

$e^?(_)\quad := \ ???$

$z_1 \oplus^? z_2 := \ ???$

[Giesl]

Calculating with Functions

[Bird, Burstall & Darlington, Meijer, ...]

$$\begin{aligned}
&\texttt{rev}(xs_1 \mathbin{+\!\!+} xs_2) \\
&\quad = \texttt{rev++}(xs_1, xs_2) \\
&\quad = \texttt{rev++'}(xs_1) \oplus^{?} e^{?}(xs_2) \\
&\quad = \mathbf{rev}(xs_2) \mathbin{+\!\!+} \texttt{rev++'}(xs_1)
\end{aligned}$$

solution for accumulator removal

$$e^{?}(_) := \mathbf{rev}(_)$$

$$z_1 \oplus^{?} z_2 := \mathbf{z_2 \mathbin{+\!\!+} z_1}$$

Calculating with Functions

[Bird, Burstall & Darlington, Meijer, ...]

$$
\begin{aligned}
&\mathtt{rev}(\mathtt{xs}_1 \mathbin{+\!\!+} \mathtt{xs}_2) \\
&\quad = \mathtt{rev{+\!\!+}}(\mathtt{xs}_1, \mathtt{xs}_2) \\
&\quad = \mathtt{rev{+\!\!+}'}(\mathtt{xs}_1) \oplus^{?} \mathtt{e}^{?}(\mathtt{xs}_2) \\
&\quad = \mathtt{rev}(\mathtt{xs}_2) \mathbin{+\!\!+} \mathbf{rev{+\!\!+}'}(\mathtt{xs}_1) \\
&\quad = \mathtt{rev}(\mathtt{xs}_2) \mathbin{+\!\!+} \mathbf{rev}(\mathtt{xs}_1)
\end{aligned}
$$

recognize that

$\mathbf{rev{+\!\!+}'} \equiv \mathbf{rev}$

Why it works

Fusion **regularizes** computations

- optimize away intermediate results (original goal)
- `fg` like `f` but extra parameters and more complex base case
- `fg` represents a “canned” (amortized) inductive proof

Fused function rev++(xs_1, xs_2)

```
rev++(xs₁, xs₂) = match xs₁
                  | case []   → rev(xs₂)
                  | case y::ys → rev++(ys, xs₂) ++ [y]
```

base case: additional computation

like rev

extra parameter

Why it works: A.R. picks up what is left by fusion

Fusion **regularizes** computations

- optimize away intermediate results (original goal)
- `fg` like `f` but extra parameters and more complex base case
- `fg` represents a "canned" (amortized) **inductive proof**

Accumulator removal **factors** computations

- push computation fragments out of functions
- template _ $\oplus^?$ $e^?$(_) represents **inductive generalizations**
- requires choice, but strictly more powerful than fusion! [Zhu]

Accumulator Removal for $\texttt{rev++}(\texttt{xs}_1, \texttt{xs}_2)$

$$\texttt{rev++}(\texttt{xs}_1, \texttt{xs}_2) = \texttt{rev++'}(\texttt{xs}_1) \oplus^? e^?(\texttt{xs}_2)$$

avoid accumulator in recursion

compensation operator

compensation term

Accumulator Removal for $\mathbf{rev{+\!\!+}(xs_1,\ xs_2)}$

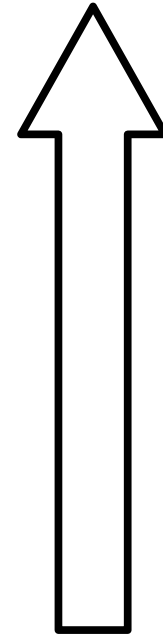
$\mathtt{rev{+\!\!+}}(xs_1,\ xs_2) = \mathbf{rev{+\!\!+}'}(xs_1) \oplus^? \mathbf{e}^?(xs_2)$

```
(match xs₁
 | case []     → rev(xs₂)
 | case y::ys  → rev++(ys, xs₂) ++ [y])
    =
(match xs₁
  | case []     → base?
  | case y::ys  → rec?(rev++'(ys), y)) ⊕? e?(xs₂)
```

Accumulator Removal for rev++(xs₁, xs₂)

```
rev++(xs₁, xs₂) = rev++'(xs₁) ⊕? e?(xs₂)

(match xs₁
 | case []     → rev(xs₂)
 | case y::ys → rev++(ys, xs₂) ++ [y])
   =
(match xs₁
  | case []     → base?
  | case y::ys → rec?(rev++'(ys), y)) ⊕? e?(xs₂)
```



additional structure to instantiate _ ⊕? e?(_)

Accumulator Removal for $\mathbf{rev{+\!\!+}(xs_1,\ xs_2)}$

```
rev++(xs₁, xs₂) = rev++'(xs₁) ⊕? e?(xs₂)

(match xs₁
  | case []     → rev(xs₂)
  | case y::ys  → rev++(ys,
    =
rev(xs₂) ++ (match xs₁
   | case []     → []
   | case y::ys  → rev++'(ys) ++ p[y])
```

```
e?(_)       := rev(_)
z₁ ⊕? z₂    := z₂ ++ z₁
base        := []
rec         := (original body)
```

Heuristics for Accumulator Removal

```
f(x, z) = f'(x) ⊕? e?(z)
```

avoid accumulator in recursion

compensation operator

compensation term

Heuristics for Accumulator Removal

```
f(x, z) = f'(x) ⊕? e?(z)
```

- avoid accumulator in recursion
- compensation operator
- compensation term

- Choose new base case of `f'` and compensation operator from functions with neutral elements (e.g. `++` with `[]` left/right)

Heuristics for Accumulator Removal

$$f(x, z) = f'(x) \oplus^{?} e^{?}(z)$$

avoid accumulator in recursion

compensation operator

compensation term

- Choose new base case of `f'` and compensation operator from functions with neutral elements (e.g. `++` with `[]` left/right)

```
             (..(rev(xs₂) ++ [y₁]) .. ++ [yₙ₋₁]) ++ [yₙ]
= rev(xs₂) ++ (..([]        ++ [y₁]) .. ++ [yₙ₋₁]) ++ [yₙ]
```

Heuristics for Accumulator Removal


```
f(x, z) = f'(x) ⊕? e?(z)
```

- avoid accumulator in recursion
- compensation operator
- compensation term

- Choose new base case of `f'` and compensation operator from functions with neutral elements (e.g. `++` with `[]` left/right)
- Choose compensation term as original base case

  (side conditions apply)

- Choose new recursive case(s) of `f'` unchanged

  (strong limitation: misses some results)

Heuristics for Accumulator Removal

```
f(x, z) = f'(x) ⊕? e?(z)
```

avoid accumulator in recursion

compensation operator

compensation term

- Choose new base case of `f'` and compensation operator from functions with neutral elements (e.g. `++` with `[]` left/right)
- Choose compensation term as original base case

  (side conditions apply)

- Choose new recursive case(s) of `f'` unchanged

  (strong limitation: misses some results)

Fusing $\mathrm{rev}(xs_1 \mathbin{+\!\!+} xs_2)$

```
rev++(xs₁, xs₂) := rev(xs₁ ++ xs₂)
        = rev(match xs₁
                | case []     → xs₂
                | case y::ys  → y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys  → rev(y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys  → rev(ys ++ xs₂) ++ [y]
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys  → rev++(ys, xs₂) ++ [y]
```

Define and Unfold [Bird, Burstall & Darlington]

```
rev++(xs₁, xs₂) := rev(xs₁ ++ xs₂)
        = rev(match xs₁
                | case []     → xs₂
                | case y::ys → y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(ys ++ xs₂) ++ [y]
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev++(ys, xs₂) ++ [y]
```

Shift Function Application [Turchin, …]

```
rev++(xs₁, xs₂) := rev(xs₁ ++ xs₂)
        = rev(match xs₁
                | case []     → xs₂
                | case y::ys → y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(ys ++ xs₂) ++ [y]
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev++(ys, xs₂) ++ [y]
```

Apply **Definitions** and known Lemmas

[Gill, Launchbury, SPJ]

```
rev++(xs₁, xs₂) := rev(xs₁ ++ xs₂)
        = rev(match xs₁
                | case []     → xs₂
                | case y::ys → y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(ys ++ xs₂) ++ [y]
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev++(ys, xs₂) ++ [y]
```

Simplify and **Fold** [Bird, Burstall & Darlington]

```
rev++(xs₁, xs₂) := rev(xs₁ ++ xs₂)
        = rev(match xs₁
                | case []     → xs₂
                | case y::ys → y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(y :: (ys ++ xs₂))
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev(ys ++ xs₂) ++ [y]
        = match xs₁
            | case []     → rev(xs₂)
            | case y::ys → rev++(ys, xs₂) ++ [y]
```

Fusing **rev(rev(xs))**

```
rev2(xs) ≔ rev(rev(xs))
        = rev(match xs
                | case [] → []
                | case y::ys → rev(ys) ++ [y])
        = match xs
            | case [] → rev([])
            | case y::ys → rev(rev(ys) ++ [y])
        = match xs
            | case [] → rev([])
            | case y::ys → rev([y]) ++ rev(rev(ys))
        = match xs
            | case [] → []
            | case y::ys → y :: rev2(ys)
        = id(xs) = xs
```

Final Lemma

```
rev2(xs) := rev(rev(xs))
      = rev(match xs
            | case [] → []
            | case y::ys → rev(ys) ++ [y])
      = match xs
         | case [] → rev([])
         | case y::ys → rev(rev(ys) ++ [y])
      = match xs
         | case [] → rev([])
         | case y::ys → rev([y]) ++ rev(rev(ys))
      = match xs
         | case [] → []
         | case y::ys → y :: rev2(ys)
      = id(xs) = xs
```

Define and Unfold [Bird, Burstall & Darlington]

```
rev2(xs) := rev(rev(xs))
       = rev(match xs
              | case [] → []
              | case y::ys → rev(ys) ++ [y])
       = match xs
          | case [] → rev([])
          | case y::ys → rev(rev(ys) ++ [y])
       = match xs
          | case [] → rev([])
          | case y::ys → rev([y]) ++ rev(rev(ys))
       = match xs
          | case [] → []
          | case y::ys → y :: rev2(ys)
       = id(xs) = xs
```

Shift Function Application [Turchin, ...]

```
rev2(xs) := rev(rev(xs))
      = rev(match xs
              | case [] → []
              | case y::ys → rev(ys) ++ [y])
      = match xs
          | case [] → rev([])
          | case y::ys → rev(rev(ys) ++ [y])
      = match xs
          | case [] → rev([])
          | case y::ys → rev([y]) ++ rev(rev(ys))
      = match xs
          | case [] → []
          | case y::ys → y :: rev2(ys)
      = id(xs) = xs
```

Intermittent Deduction using Prior Lemmas

[Gill, Launchbury, SPJ]

```
rev2(xs) := rev(rev(xs))
       = rev(match xs
              | case [] → []
              | case y::ys → rev(ys) ++ [y])
       = match xs
           | case [] → rev([])
           | case y::ys → rev(rev(ys) ++ [y])
       = match xs
           | case [] → rev([])
           | case y::ys → rev([y]) ++ rev(rev(ys))
       = match xs
           | case [] → []
           | case y::ys → y :: rev2(ys)
       = id(xs) = xs
```

Intermittent Deduction using Prior Lemmas

[Gill, Launchbury, SPJ]

```
rev2(xs) := rev(rev(xs))
       = rev(match xs
             | case [] → []
             | case y::ys → rev(ys) ++ [y])
       = match xs
          | case [] → rev([])
          | case y::ys → rev(rev(ys) ++ [y])
       = match xs
          | case [] → rev([])
          | case y::ys → rev([y]) ++ rev(rev(ys))
```

most difficult step, requires to know already

$\texttt{rev}(xs_1 \mathbin{+\!\!+} xs_2) = \texttt{rev}(xs_2) \mathbin{+\!\!+} \texttt{rev}(xs_1)$

Simplify and Fold [Bird, Burstall & Darlington]

```
rev2(xs) := rev(rev(xs))
      = rev(match xs
            | case [] → []
            | case y::ys → rev(ys) ++ [y])
      = match xs
        | case [] → rev([])
        | case y::ys → rev(rev(ys) ++ [y])
      = match xs
        | case [] → rev([])
        | case y::ys → rev([y]) ++ rev(rev(ys))
      = match xs
        | case [] → []
        | case y::ys → y :: rev2(ys)
      = id(xs) = xs
```

Recognize identity, constant, existing Functions

```
rev2(xs) := rev(rev(xs))
        = rev(match xs
                | case [] → []
                | case y::ys → rev(ys) ++ [y])
        = match xs
            | case [] → rev([])
            | case y::ys → rev(rev(ys) ++ [y])
        = match xs
            | case [] → rev([])
            | case y::ys → rev([y]) ++ rev(rev(ys))
        = match xs
            | case [] → []
            | case y::ys → y :: rev2(ys)
        = id(xs) = xs
```