When Separation Arithmetic is Enough

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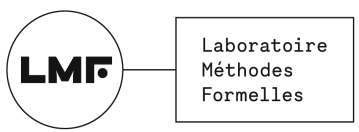
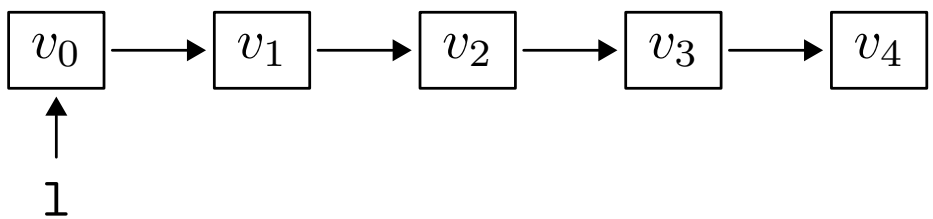


The good specification is the one that is easy to understand.

The good invariant is the one that is easy to verify.

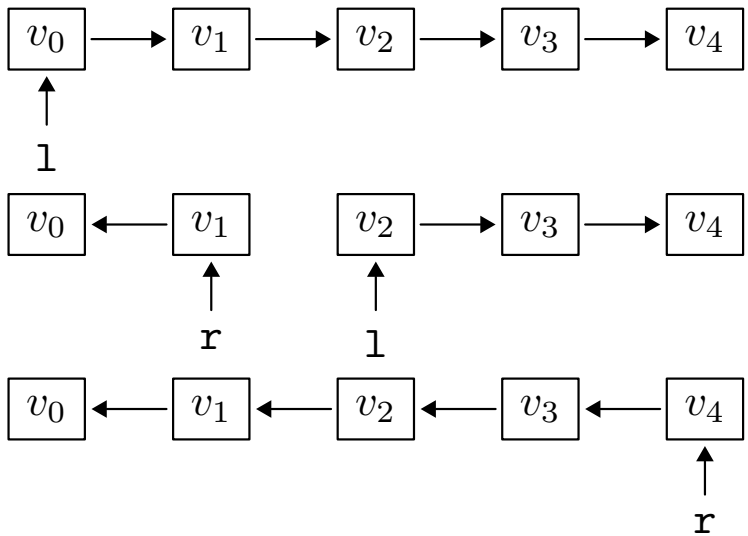
1. warm up: classic list reversal
2. reversing values in constant space
3. binary tree traversal using Morris's algorithm
4. other examples
5. discussion

```
type lst =
  | Nil
  | Cons of { mutable car: elt; mutable cdr: lst }
```



list reversal

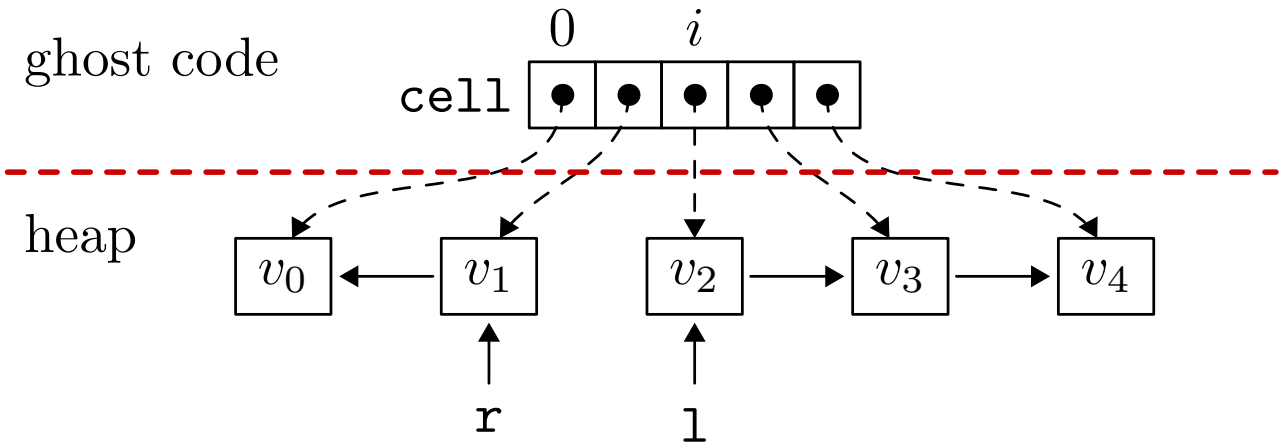
```
let rec aux r l = match l with
  | Nil     -> r
  | Cons c -> let n = c.cdr in
              c.cdr <- r;
              aux l n

let list_reversal (l: lst) : lst =
  aux Nil l
```

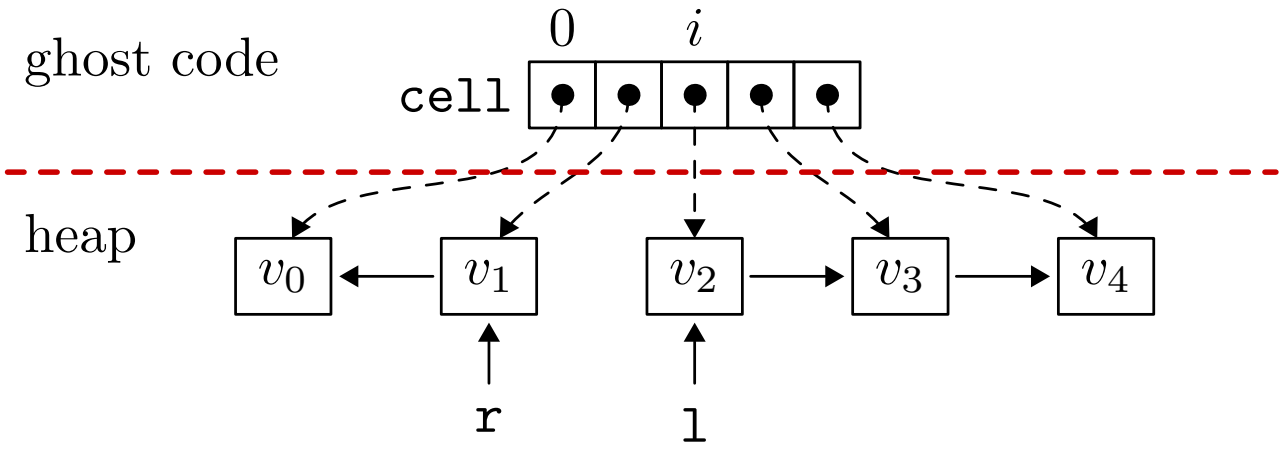


Why3 model of the heap

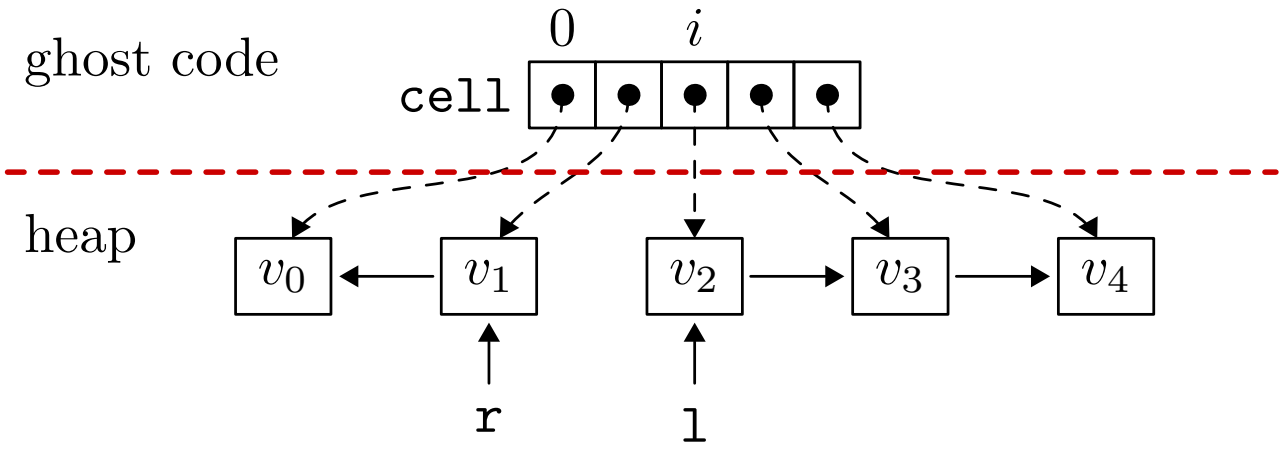
```
type elt
type lst

constant nil: lst

type mem = { mutable mcar: lst -> elt;
             mutable mcdr: lst -> lst }

val mem: mem
```

(this is Burstall's component-as-array principle)


ghost code
0
i
cell
heap
v_0
v_1
v_2
v_3
v_4
r
l




```
type list_shape = { cell: int -> lst; size: int }
```




```
type list_shape = { cell: int -> lst; size: int }
```

```
let list_reversal (ghost ls: list_shape)
                  (l: lst) : (r: lst)
  = ...
```

the cells are non-nil and pairwise distinct:

```
type list_shape = { cell: int -> lst; size: int }
invariant
  { 0 <= size }
invariant
  { forall i. 0 <= i <  size -> cell[i] <> nil }
invariant
  { forall i. 0 <= i <  size ->
    forall j. 0 <= j <= size -> i <> j ->
    cell[i] <> cell[j] }
```

they relate a given list shape to the contents of the heap

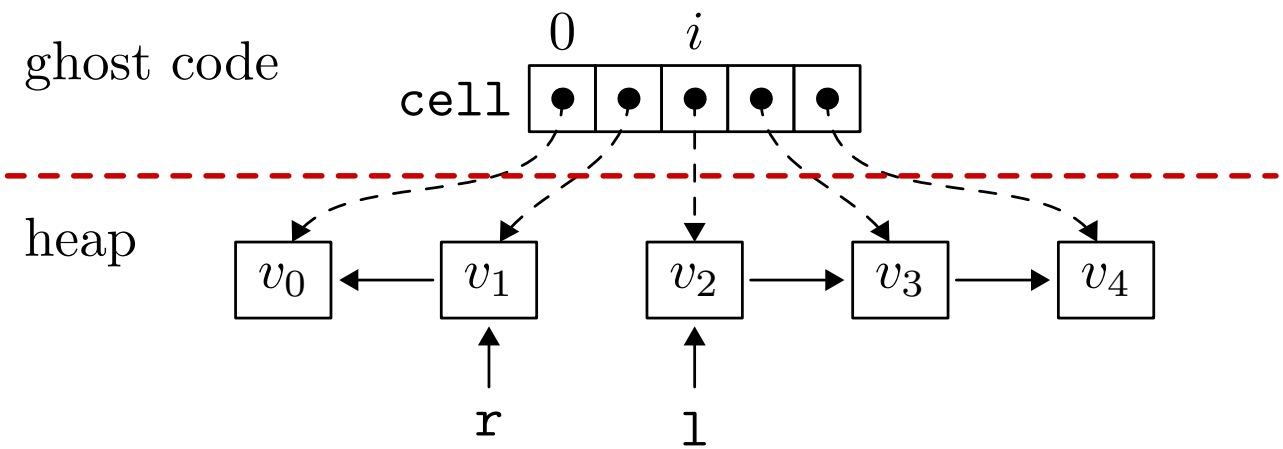
e.g. there is a list from `p` to `q` corresponding to `cell[lo..hi]`

```
predicate listLR (ls: list_shape) (m: mem)
                 (p: lst) (lo hi: int) (q: lst) =
  0 <= lo <= hi <= ls.size /\
  p = ls.cell[lo] /\
  q = ls.cell[hi] /\
  forall i. lo <= i < hi ->
    m.mcdr[ls.cell[i]] = ls.cell[i+1]
```

the key: this is not recursive!

similarly for `listRL`

```
let rec aux (ghost ls: list_shape)
            (ghost i: int) (r l: lst) : lst
  requires { listLR ls mem l i ls.size nil }
  requires { listRL ls mem r i }
  variant  { ls.size - i }
  ensures  { listRL ls mem result ls.size  }
```



the verification is fully automatic

no lemma, no assertion in the code, no user interaction

how to build the list shape in the first place?

```
let ghost shape_of_list (p q: lst) : (ls: list_shape)
  requires { linked_list mem p q }
  ensures  { listLR ls mem p 0 ls.size q }
```

use your favorite definition for `linked_list` here

reversing values in constant space

what if we want to reverse a list by swapping the `car` fields instead of reversing the `cdr` fields?

can we do this in linear time and constant space?

1. traverse the list at speed 1 and 2 (tortoise and hare) to find its middle
   - reverse the first half as you go
2. traverse the two halves simultaneously, swapping the `cars`
   - restore the first half as you go

illustration with $n = 4$

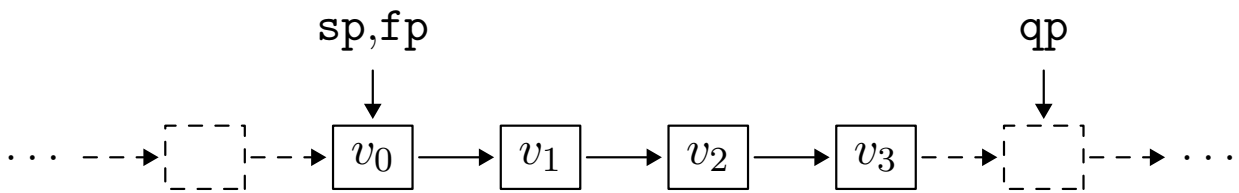


illustration with $n = 4$

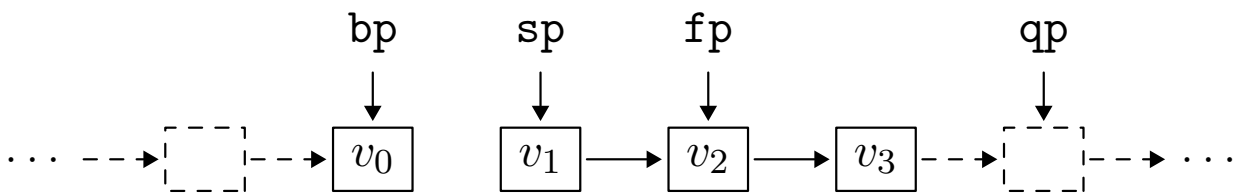


illustration with $n = 4$

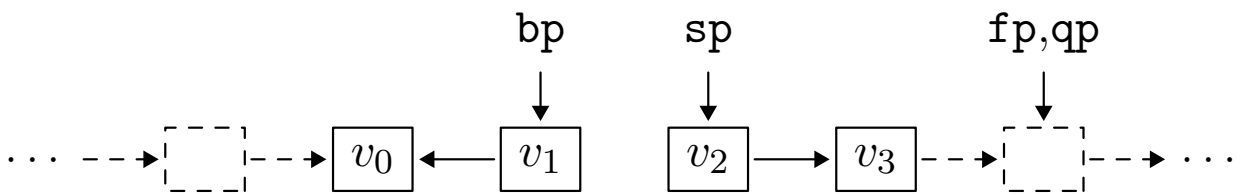


illustration with $n = 4$

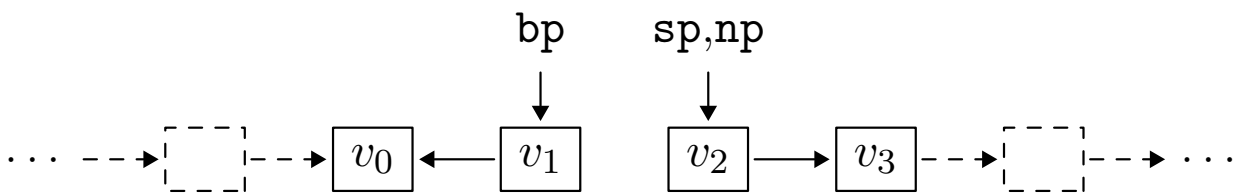


illustration with $n = 4$

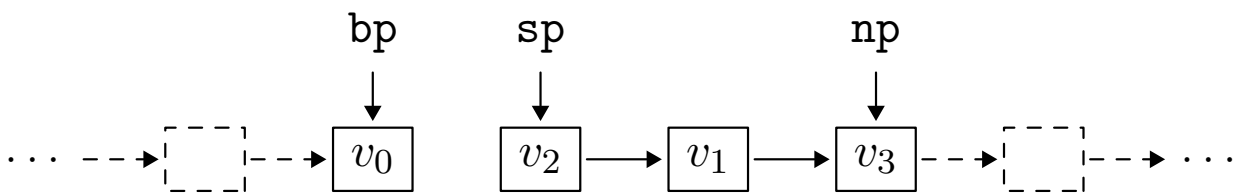


illustration with $n = 4$

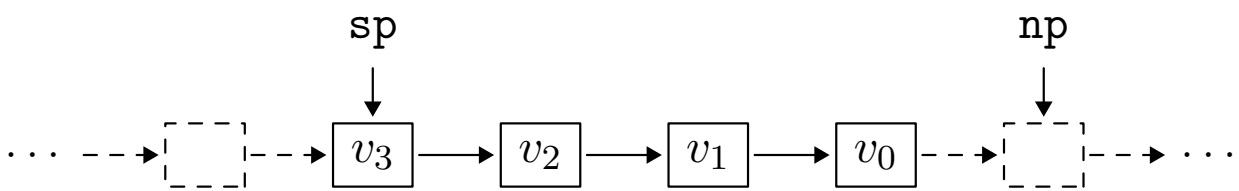


illustration with $n = 5$

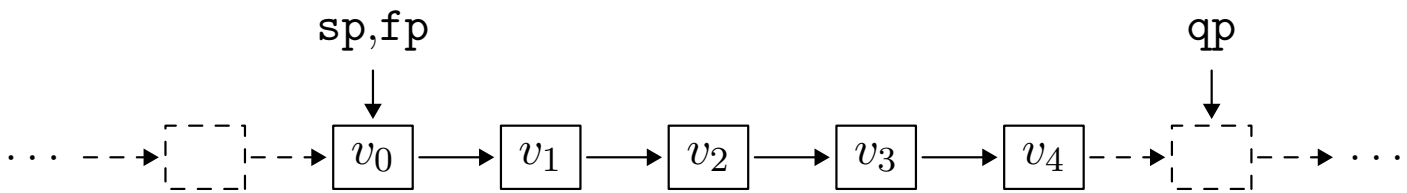


illustration with $n = 5$

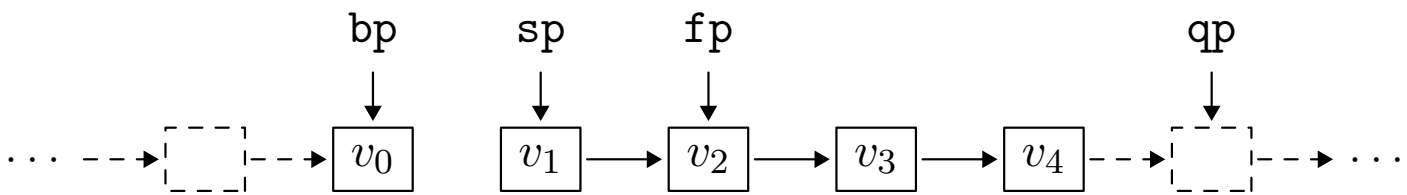


illustration with $n = 5$

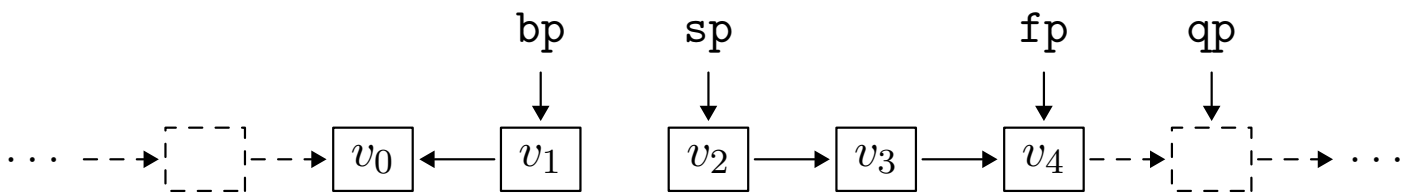


illustration with $n = 5$

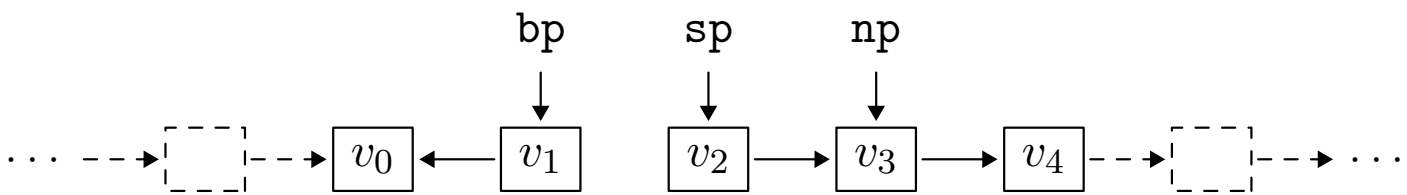


illustration with $n = 5$

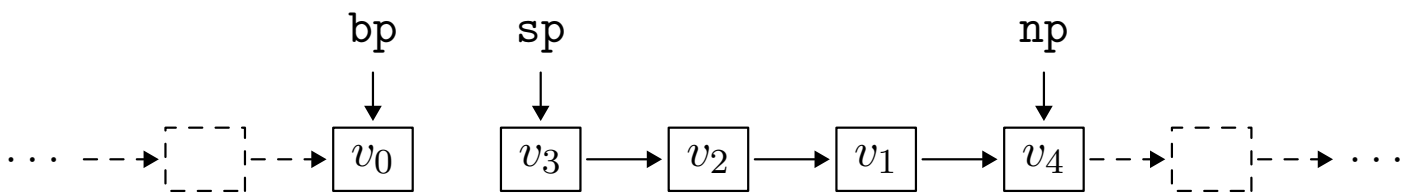
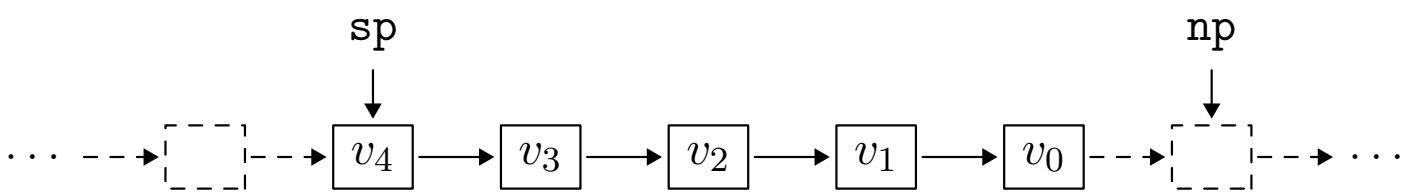
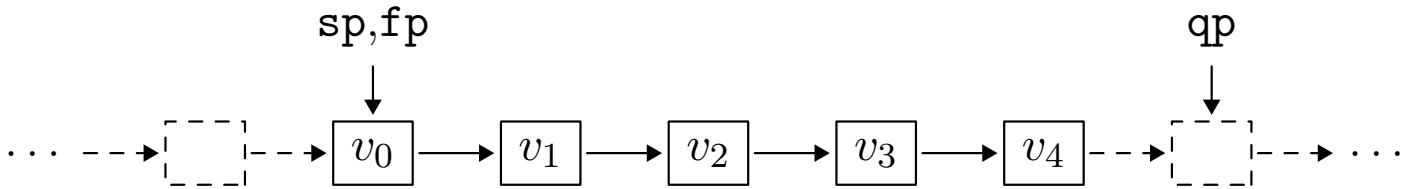


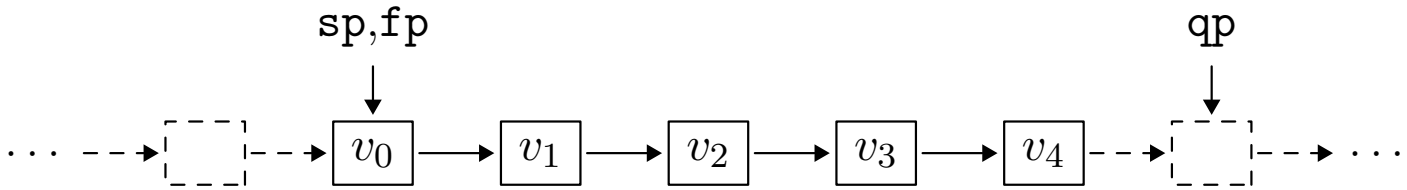
illustration with $n = 5$



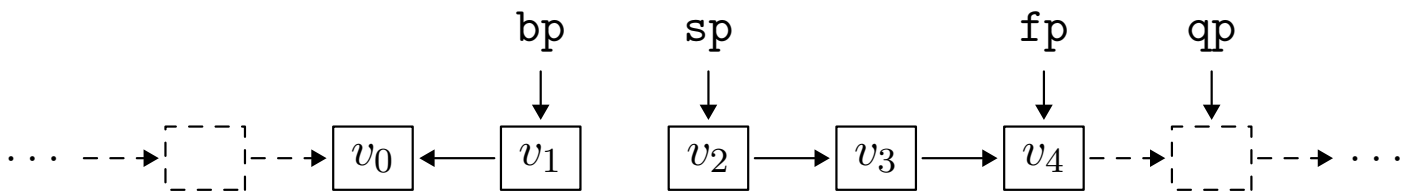
```
let value_reverse (sp: lst) (qp: lst) : unit =
  tortoise_hare Nil sp sp qp
```



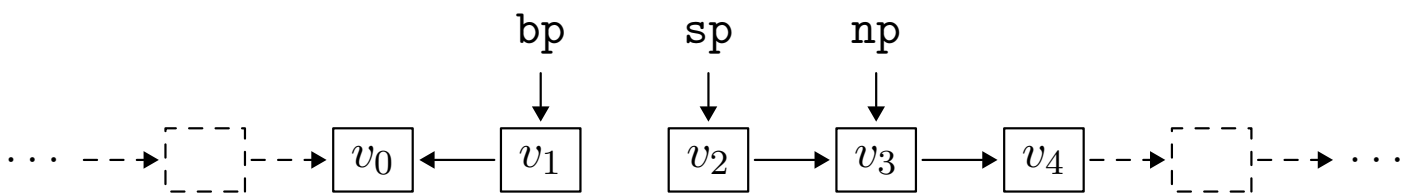
```
let value_reverse (sp: lst) (qp: lst) : unit =
  tortoise_hare Nil sp sp qp
```




```
let rec tortoise_hare bp sp fp qp = match sp, fp with
  | _ when fp == qp ->
    back_again bp sp sp
  | Cons sc, Cons {cdr = nfp} when nfp == qp ->
    back_again bp sp sc.cdr
  | Cons sc, Cons {cdr = Cons {cdr = nfp}} ->
    let nsp = sc.cdr in sc.cdr <- bp;
    tortoise_hare sp nsp nfp qp
```



```
let rec back_again bp sp np = match bp, np with
  | Cons bc, Cons nc ->
      let tmp = bc.car in
      bc.car <- nc.car; nc.car <- tmp;
      let nbp = bc.cdr in bc.cdr <- sp;
      back_again nbp bp nc.cdr
  | _ -> ()
```



time complexity is linear (two traversals)

all calls are tail calls, so the space complexity is constant
(could be written with loops)

we reuse the same idea, and the very same `list_shape`

```
let value_reverse (ghost ls: list_shape) (sp qp: lst)
  requires { listLR ls mem sp 0 ls.size qp }
  ensures  { forall i. 0 <= i < ls.size ->
                mem.mcar[ls.cell[i]]
                  = old mem.mcar[ls.cell[ls.size-1-i]]
              /\ mem.mcdr[ls.cell[i]]
                  = old mem.mcdr[ls.cell[i]] }
```

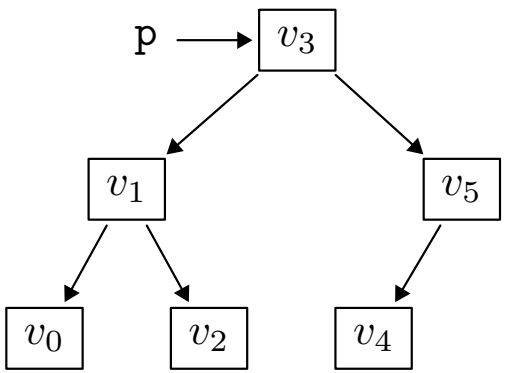
again, the verification is fully automatic

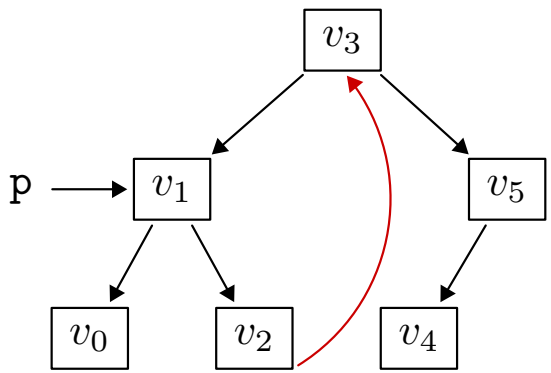
binary tree traversal using Morris's algorithm

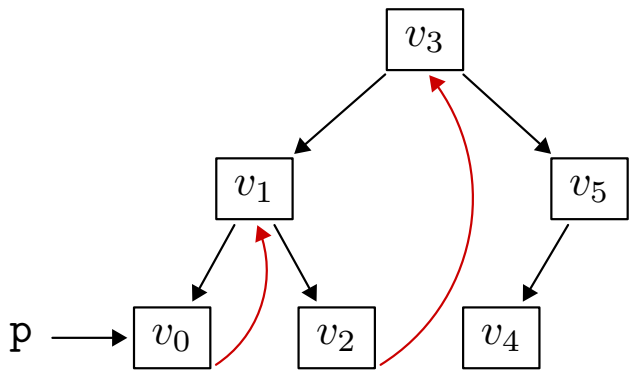
perform an in-order traversal of a binary tree in constant space and linear time

solution: mutate the tree as you go, to remember what to do next, and restore it before completion

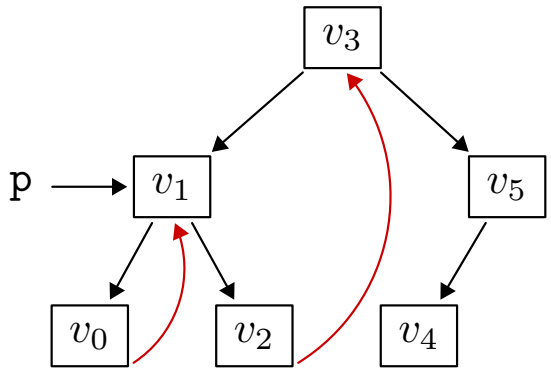
Joseph M. Morris, Traversing Binary Trees Simply and Cheaply.
Information Processing Letters 9(5), 197–200 (1979)


p
v_3
v_1
v_5
v_0
v_2
v_4

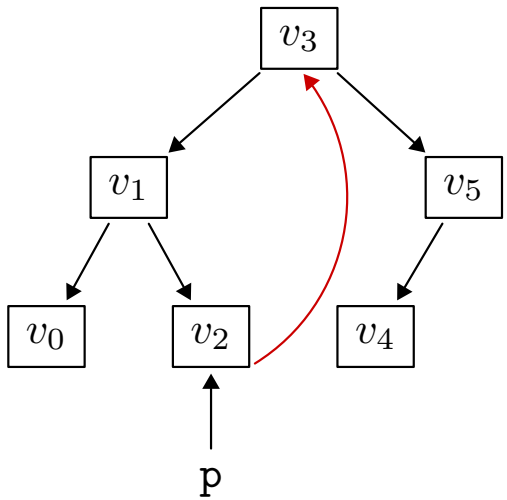

v_3
p
v_1
v_5
v_0
v_2
v_4



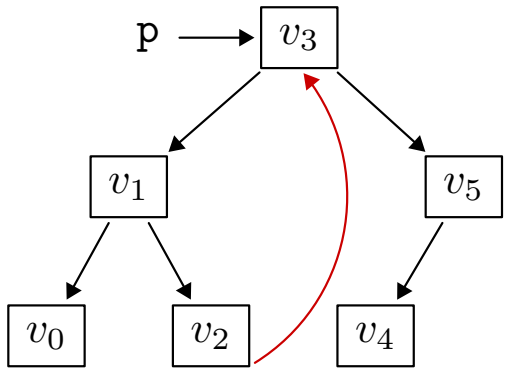
visit v_0



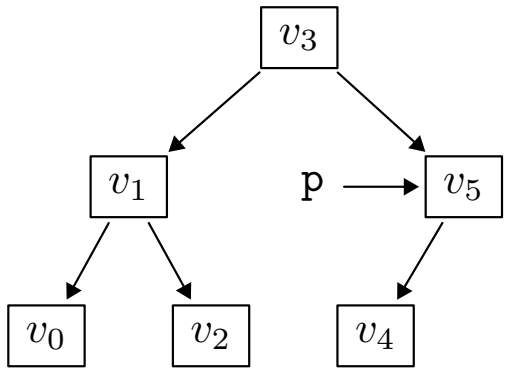
visit v_0



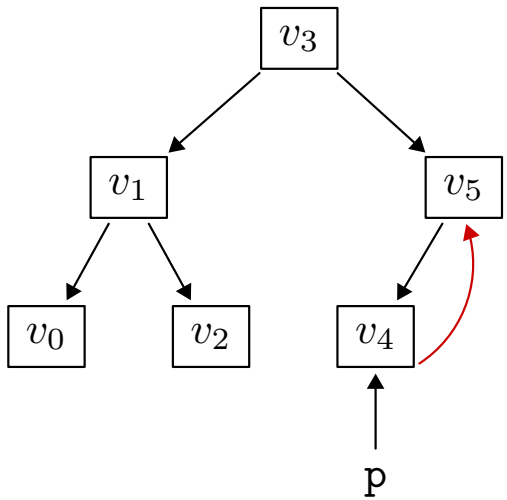
visit v_0, visit v_1



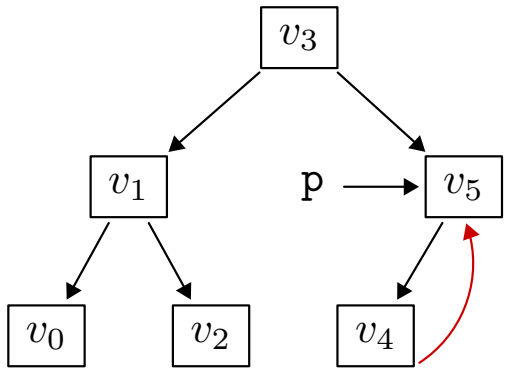
visit v_0, visit v_1, visit v_2



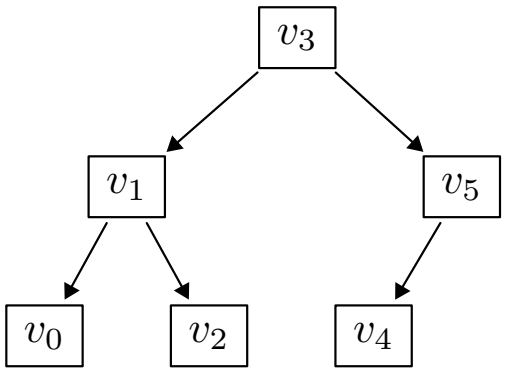
visit v_0, visit v_1, visit v_2, visit v_3



visit v_0, visit v_1, visit v_2, visit v_3



visit v_0, visit v_1, visit v_2, visit v_3, visit v_4



visit v_0, visit v_1, visit v_2, visit v_3, visit v_4, visit v_5

```
type tree =
  | E
  | N of { mutable  left: tree;
           mutable   dat: elt;
           mutable right: tree; }
```

```
let traversal (visit: elt -> unit) (p: tree) : unit =
  if p != E then morris visit p
```

Morris's algorithm

```
let rec morris visit (N {left; dat; right} as p) =
  if left != E && warp p left then
    morris visit left
  else (
    visit dat;
    if right != E then morris visit right )
```

```
let traversal (visit: elt -> unit) (p: tree) : unit =
  if p != E then morris visit p
```

Morris's algorithm

```
let rec warp p (N q) =
  if q.right == E then (q.right <- p; true ) else
  if q.right == p then (q.right <- E; false) else
  warp p q.right
```

```
let rec morris visit (N {left; dat; right} as p) =
  if left != E && warp p left then
    morris visit left
  else (
    visit dat;
    if right != E then morris visit right )
```

```
let traversal (visit: elt -> unit) (p: tree) : unit =
  if p != E then morris visit p
```

the space complexity is constant (all calls are tail calls)

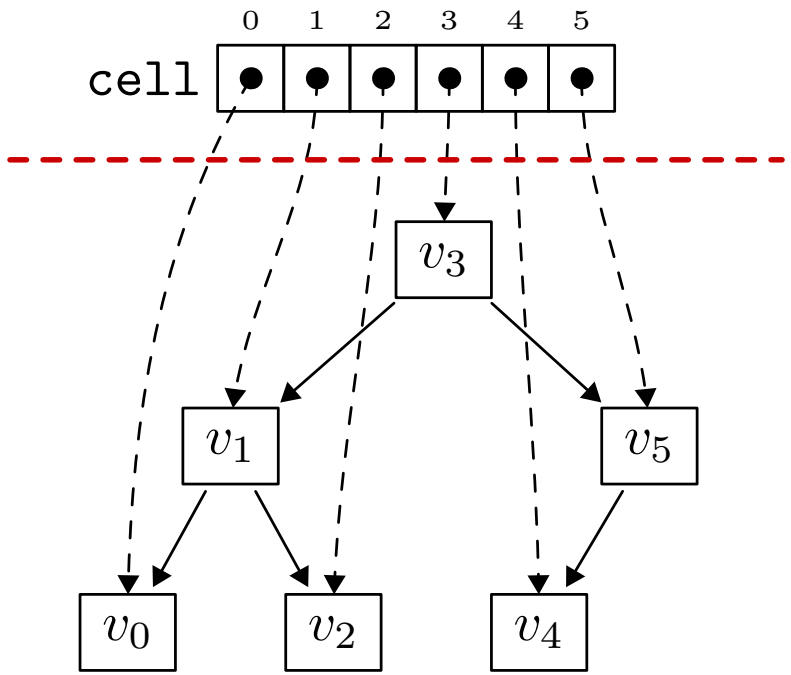
the time complexity is linear! (exercise: show it)

let us verify this with Why3, using separation arithmetic

let us verify this with Why3, using separation arithmetic

```
type tree_shape = {
  cell: int -> tree; (* tree nodes *)
  size: int;
  lchd: int -> int;  (* left child *)
  rchd: int -> int;  (* right child *)
  lmdt: int -> int;  (* leftmost descendant *)
  rmdt: int -> int;  (* rightmost descendant *)
}
```

let us verify this with Why3, using separation arithmetic



	0	1	2	3	4	5
lchd	0	0	2	1	4	4
rchd	0	2	2	5	4	5
lmdt	0	0	2	0	4	4
rmdt	0	2	2	5	4	5

cells are non-empty and pairwise distinct

```
invariant
  { 0 <= size }
invariant
  { forall i. 0 <= i < size -> cell[i] <> empty }
invariant
  { forall i. 0 <= i < size ->
    forall j. 0 <= j < size -> i <> j ->
    cell[i] <> cell[j] }
```

(as for `list_shape`)

cells are listed according to the in-order traversal

```
invariant
  { forall i. 0 <= i < size ->
    0 <= lmdt[i] =  lmdt[lchd[i]] <= lchd[i] <= i
 /\ i <= rchd[i] <= rmdt[rchd[i]] =  rmdt[i] <  size }
```

finally, here are no spurious cells

```
invariant
  { forall i. 0 <= i < size ->
    (if lchd[i] = i then lmdt[i]        = i
                    else rmdt[lchd[i]] = i-1)
 /\ (if rchd[i] = i then rmdt[i]        = i
                    else lmdt[rchd[i]] = i+1) }
```

as for lists, we model the heap

```
type mem = ...
```

and we related the heap and the tree shape with predicates

```
predicate wf_lhs (t: tree_shape) (m: mem) (lo hi: int)
  = ...
predicate wf_rhs (t: tree_shape) (m: mem) (lo hi: int)
  = ...
predicate warped (t: tree_shape) (m: mem) (lo hi: int)
  = ...
```

no recursion, only universal quantifiers and linear arithmetic

visited nodes are appended to a ghost `trace`

and we show that it coincides with `cell`

```
let traversal (ghost t: tree_shape)
              (ghost k: int) (p: loc) : unit
  ...
  ensures { trace.length = t.size }
  ensures { forall i. 0 <= i < t.size ->
              trace[i] = t.cell[i] }
```

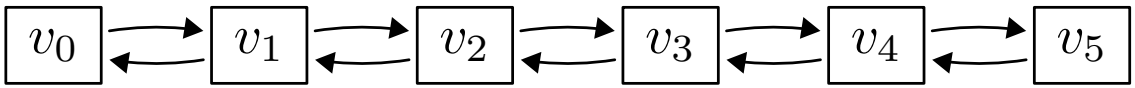
this time, the verification is slightly more complex

- need for one lemma
- a little bit of user interaction

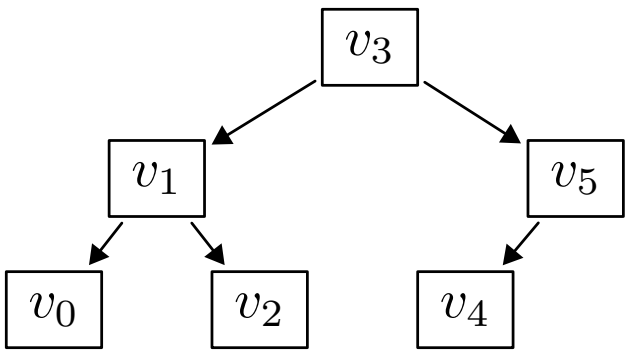
yet mostly automatic

other examples

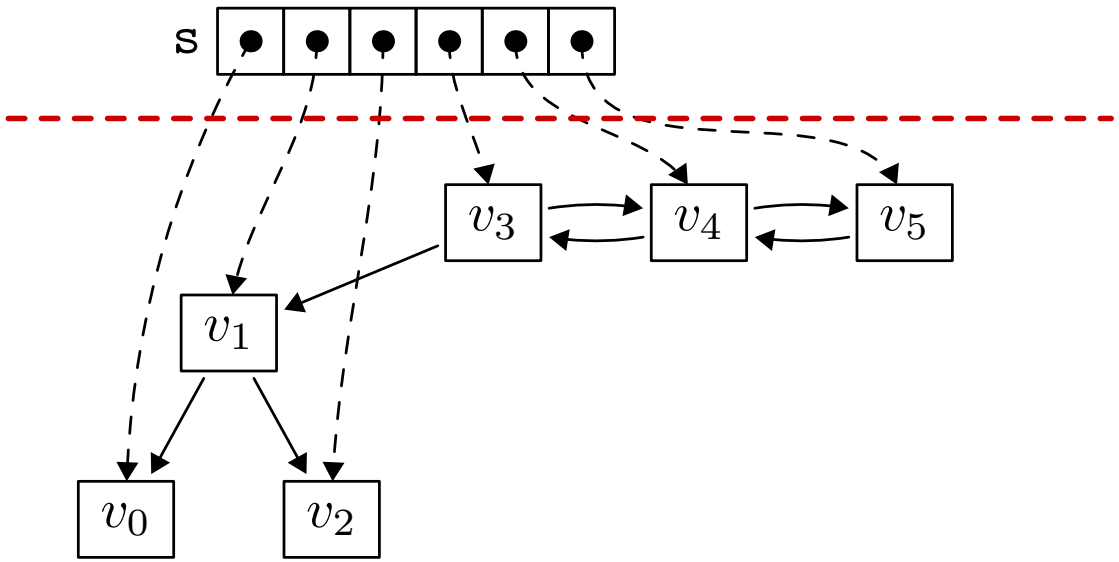
verify a program that turns a doubly-linked list



into a balanced binary tree



(`prev` and `next` pointers are now `left` and `right` pointers)


s
v_3
v_4
v_5
v_1
v_0
v_2

verify a program that removes the node with the minimal key in a binary search tree

we believe that separation arithmetic could also be used successfully on this example (yet to be done, though)

discussion

of course, Separation Logic is a great tool to reason about pointer-based data structures

$$\begin{array}{rcl} \mathit{list}\ \varepsilon\ p & \stackrel{\text{def}}{=} & p = \mathit{null} \\ \mathit{list}\ (v \cdot \ell)\ p & \stackrel{\text{def}}{=} & \exists q.\ p \mapsto (v, q) \star \mathit{list}\ \ell\ q \end{array}$$

separation arithmetic cannot be compared, but is a powerful tool when applicable

the idea of using local invariants based on universal quantification rather than recursive definitions is folklore

e.g.
JCF, Simpler Proofs with Decentralized Invariants.
Journal of Logical and Algebraic Methods in Programming, 2021

Schorr-Waite algorithm performs a DFS of a graph, using the graph structure itself to encode the stack (at the price of a small extra space in each node)

it was once used as a benchmark for program verifiers

open question:
can we use separation arithmetic for something like Schorr-Waite?